

List $(d,1)$ -total labelling of graphs embedded in surfaces*

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Abstract

The $(d,1)$ -total labelling of graphs was introduced by Havet and Yu. In this paper, we consider the list version of $(d,1)$ -total labelling of graphs. Let G be a graph embedded in a surface with Euler characteristic ε whose maximum degree $\Delta(G)$ is sufficiently large. We prove that the $(d,1)$ -total choosability $C_{d,1}^T(G)$ of G is at most $\Delta(G) + 2d$.

Keywords: $(d,1)$ -total labelling; list $(d,1)$ -total labelling; $(d,1)$ -total choosability; graphs

MSC: 05C15

1 Introduction

In this paper, graph G is a simple connected graph with a finite vertex set $V(G)$ and a finite edge set $E(G)$. If X is a set, we usually denote the cardinality of X by $|X|$. Denote the set of vertices adjacent to v by $N(v)$. The degree of a vertex v in G , denoted by $d_G(v)$, is the number of edges incident with v . We sometimes write $V, E, d(v), \Delta, \delta$ instead of $V(G), E(G), d_G(v), \Delta(G), \delta(G)$, respectively. Let G be a plane graph. We always denote by $F(G)$ the face set of G . The degree of a face f , denoted by $d(f)$, is the number of edges incident with it, where cut edge is counted twice. A k -, k^+ - and k^- -vertex (or face) in graph G is a vertex (or face) of degree k , at least k and at most k , respectively.

The $(d,1)$ -total labelling of graphs was introduced by Havet and Yu [4]. A k -($d,1$)-total labelling of a graph G is a function c from $V(G) \cup E(G)$ to the color set $\{0, 1, \dots, k\}$ such that $c(u) \neq c(v)$ if $uv \in E(G)$, $c(e) \neq c(e')$ if e and e' are two adjacent edges, and $|c(u) - c(e)| \geq d$ if vertex u is incident to the edge e . The minimum k such that G has a k -($d,1$)-total labelling is called the $(d,1)$ -total labelling number and denoted by $\lambda_d^T(G)$. Readers are referred to [1, 3, 5, 6, 7] for further research.

Suppose that $L(x)$ is a list of colors available to choose for each element $x \in V(G) \cup E(G)$. If G has a $(d,1)$ -total labelling c such that $c(x) \in L(x)$ for all $x \in V(G) \cup E(G)$, then we say that c is an L -($d,1$)-total labelling of G , and G is L -($d,1$)-total labelable (sometimes we also say G is list $(d,1)$ -total labelable). Furthermore, if G is L -($d,1$)-total labelable for any L with $|L(x)| = k$ for each $x \in V(G) \cup E(G)$, we say that

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G is k -($d,1$)-total choosable. The $(d,1)$ -total choosability, denoted by $C_{d,1}^T(G)$, is the minimum k such that G is k -($d,1$)-total choosable. Actually, when $d = 1$, the list $(1,1)$ -total labelling is the well-known list total coloring of graphs. It is known that for list version of total colorings there is a list total coloring conjecture (LTCC). Therefore, it is natural to conjecture that $C_{d,1}^T(G) = \lambda_d^T(G) + 1$. Unfortunately, counterexamples that $C_{d,1}^T(G)$ is strictly greater than $\lambda_d^T(G) + 1$ can be found in [9]. Although we can not present a conjecture like LTCC, we conjecture that $C_{d,1}^T(G) \leq \Delta + 2d$ for any graph G . In [9], we studied the list $(d,1)$ -total labelling of special graphs such as paths, trees, stars and outerplanar graphs which lend positive support to our conjecture.

In this paper, we prove that, for graphs which can be embedded in a surface with Euler characteristic ε , the conjecture is still true when the maximum degree is sufficiently large. Our main results are the following:

Theorem 1.1. *Let G be a graph embedded in a surface of Euler characteristic $\varepsilon \leq 0$ and $\Delta(G) \geq \frac{d}{2d-1} \left(10d - 8 + \sqrt{(10d-2)^2 - 24(2d-1)\varepsilon} \right) + 1$, where $d \geq 2$. Then $C_{d,1}^T(G) \leq \Delta(G) + 2d$.*

Theorem 1.2. *Let G be a graph embedded in a surface of Euler characteristic $\varepsilon > 0$. If $\Delta(G) \geq 5d + 2$ where $d \geq 2$, then $C_{d,1}^T(G) \leq \Delta(G) + 2d$.*

We prove two conclusions which are slightly stronger than the theorems above as follows.

Theorem 1.3. *Let G be a graph embedded in a surface of Euler characteristic $\varepsilon \leq 0$ and let $M \geq \frac{d}{2d-1} \left(10d - 8 + \sqrt{(10d-2)^2 - 24(2d-1)\varepsilon} \right) + 1$ where $d \geq 2$. If $\Delta(G) \leq M$, then $C_{d,1}^T(G) \leq M + 2d$. In particular, $C_{d,1}^T(G) \leq \Delta(G) + 2d$ if $\Delta(G) = M$.*

Theorem 1.4. *Let G be a graph embedded in a surface of Euler characteristic $\varepsilon > 0$ and let $M \geq 5d + 2$ where $d \geq 2$. If $\Delta(G) \leq M$, then $C_{d,1}^T(G) \leq M + 2d$. In particular, $C_{d,1}^T(G) \leq \Delta(G) + 2d$ if $\Delta(G) = M$.*

The interesting cases of Theorem 1.3 and Theorem 1.4 are when $M = \Delta(G)$. Indeed, Theorem 1.3 and Theorem 1.4 are only a technical strengthening of Theorem 1.1 and Theorem 1.2, respectively. But without them we would get complications when considering a subgraph $H \subset G$ such that $\Delta(H) < \Delta(G)$.

In Section 2, we prove some lemmas. In Section 3, we complete our main proof with discharging method.

2 Structural properties

From now on, we will use without distinction the terms *colors* and *labels*. Let c be a partial list $(d,1)$ -total labelling of G . We denote by $A(x)$ the set of colors which are still available for coloring element x of G with the partial list $(d,1)$ -total labelling c . Let G be a minimal counterexample in terms of $|V(G)| + |E(G)|$ to Theorem 1.3 or Theorem 1.4.

Lemma 2.1. *G is connected.*

Proof. Suppose that G is not connected. Without loss of generality, let G_1 be one component of G and $G_2 = G \setminus G_1$. By the minimality of G , G_1 and G_2 are both $(M + 2d)$ -($d,1$)-total choosable which implies G is $(M + 2d)$ -($d,1$)-total choosable, a contradiction. ■

Lemma 2.2. *For each $e = uv \in E(G)$, $d(u) + d(v) \geq M - 2d + 4$.*

Proof. If for some $e = uv \in E(G)$, $d(u) + d(v) \leq M - 2d + 3$. By the minimality of G , $G - e$ is $(M + 2d)$ -($d, 1$)-total choosable. We denote this coloring by c . Since $|A(e)| \geq M + 2d - (d(u) + d(v) - 2) - 2(2d - 1) \geq M + 2d - (M - 2d + 1) - 2(2d - 1) \geq 1$ under the coloring c , we can extend c to G , a contradiction. ■

Lemma 2.3. *For any edge $e = uv \in E(G)$ with $\min\{d(u), d(v)\} \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor$, we have $d(u) + d(v) \geq M + 3$.*

Proof. Suppose there is some $e = uv \in E(G)$ such that $d(u) \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor$ and $d(u) + d(v) \leq M + 2$. By the minimality of G , $G - e$ is $(M + 2d)$ -($d, 1$)-total choosable. Erase the color of vertex u , and let c be the partial list $(d, 1)$ -total labelling with $|L| = M + 2d$. Then $|A(e)| \geq M + 2d - (d(u) + d(v) - 2) - (2d - 1) \geq M + 2d - M - (2d - 1) \geq 1$ which implies that e can be properly colored. Next, for vertex u , $|A(u)| \geq M + 2d - (d(u) + (2d - 1)d(u)) \geq M + 2d - (M + 2d - 1) \geq 1$. Thus we extend the coloring c to G , a contradiction. ■

Lemma 2.4. ([2]) *A bipartite graph G is edge f -choosable where $f(uv) = \max\{d(u), d(v)\}$ for any $uv \in E(G)$.*

A k -alternator for some k ($3 \leq k \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor$) is a bipartite subgraph $B(X, Y)$ of graph G such that $d_B(x) = d_G(x) \leq k$ for each $x \in X$ and $d_B(y) \geq d_G(y) + k - M - 1$ for each $y \in Y$.

The concept of k -alternator was first introduced by Borodin, Kostochka and Woodall [2] and generalized by Wu and Wang [8].

Lemma 2.5. *There is no k -alternator $B(X, Y)$ in G for any integer k with $3 \leq k \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor$.*

Proof. Suppose that there exists a k -alternator $B(X, Y)$ in G . Obviously, X is an independent set of vertices in graph G by Lemma 2.3. By the minimality of G , we can color all elements of subgraph $G[V(G) \setminus X]$ from their lists of size $M + 2d$. We denote this partial list $(d, 1)$ -total labelling by c . Then for each edge $e = xy \in B(X, Y)$, $|A(e)| \geq M + 2d - (d_G(y) - d_B(y) + (2d - 1)) \geq M + 2d - (M - d_B(y) + (2d - 1)) \geq d_B(y)$ and $|A(e)| \geq M + 2d - (d_G(y) - d_B(y) + (2d - 1)) \geq M + 2d - (M + 2d - k) \geq k$ because $B(X, Y)$ is a k -alternator. Therefore, $|A(e)| \geq \max\{d_B(y), d_B(x)\}$. By Lemma 2.4, it follows that $E(B(X, Y))$ can be colored properly from their new color lists. Next, for each vertex $x \in X$, $|A(x)| \geq M + 2d - (d(x) + (2d - 1)d(x)) \geq M + 2d - (M + 2d - 1) \geq 1$ because $d_G(x) \leq k \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor$. Thus we extend the coloring c to G , a contradiction. ■

Lemma 2.6. *Let $X_k = \{x \in V(G) \mid d_G(x) \leq k\}$ and $Y_k = \cup_{x \in X_k} N(x)$ for any integer k with $3 \leq k \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor$. If $X_k \neq \emptyset$, then there exists a bipartite subgraph M_k of G with partite sets X_k and Y_k such that $d_{M_k}(x) = 1$ for each $x \in X_k$ and $d_{M_k}(y) \leq k - 2$ for each $y \in Y_k$.*

Proof. The proof is omitted here as it is similar with the proof of Lemma 2.4 by Wu and Wang in [8]. ■

We call y the k -master of x if $xy \in M_k$ and $x \in X_k, y \in Y_k$. By Lemma 2.3, if $uv \in E(G)$ satisfies $d(v) \leq \left\lfloor \frac{M+2d-1}{2d} \right\rfloor$ and $d(u) = M-i$, then $d(v) \geq M+3-d(u) \geq i+3$. Together with Lemma 2.6, it follows that each $(M-i)$ -vertex can be a j -master of at most $j-2$ vertices, where $3 \leq i+3 \leq j \leq \left\lfloor \frac{M+2d-1}{2d} \right\rfloor$. Each i -vertex has a j -master by Lemma 2.6, where $3 \leq i \leq j \leq \left\lfloor \frac{M+2d-1}{2d} \right\rfloor$.

3 Proof of main results

By our Lemmas above, G has structural properties in the following.

(C1) G is connected;

(C2) for each $e = uv \in E(G)$, $d(u) + d(v) \geq M - 2d + 4$;

(C3) if $e = uv \in E(G)$ and $\min\{d(u), d(v)\} \leq \left\lfloor \frac{M+2d-1}{2d} \right\rfloor$, then $d(u) + d(v) \geq M + 3$.

(C4) each i -vertex (if exists) has one j -master, where $3 \leq i \leq j \leq \left\lfloor \frac{M+2d-1}{2d} \right\rfloor$;

(C5) each $(M-i)$ -vertex (if exists) can be a j -master of at most $j-2$ vertices, where $3 \leq i+3 \leq j \leq \left\lfloor \frac{M+2d-1}{2d} \right\rfloor$.

Proof of Theorem 1.3 Let G be a minimal counterexample in terms of $|V(G)| + |E(G)|$ to Theorem 1.3. In this theorem, $M \geq \frac{d}{2d-1} \left(10d - 8 + \sqrt{(10d-2)^2 - 24(2d-1)\varepsilon} \right) + 1 \geq 10d + 1$. Thus $\left\lfloor \frac{M+2d-1}{2d} \right\rfloor \geq 6$. In the following, we apply the discharging method to complete the proof by a contradiction. At the very beginning, we assign an initial charge $w(x) = d(x) - 6$ for any $x \in V(G)$. By Euler's formula $|V| - |E| + |F| = \varepsilon$, we have $\sum_{x \in V} w(x) = \sum_{x \in V} (d(x) - 6) = -6\varepsilon - \sum_{x \in F} (2d(x) - 6) \leq -6\varepsilon$.

The discharging rule is as follows.

(R1) each i -vertex (if exists) receives charge 1 from each of its j -master, where $3 \leq i \leq j \leq 5$.

If $M \geq \Delta + 3$, then $\delta(G) \geq 6$. Otherwise, let $uv \in E(G)$ and $d(u) \leq 5$. Then $d(u) + d(v) \leq M - 3 + 5 \leq M + 2$ and $d(u) \leq \left\lfloor \frac{M+2d-1}{2d} \right\rfloor$ as $\left\lfloor \frac{M+2d-1}{2d} \right\rfloor \geq 6$, which is a contradiction to (C3). This obviously contradicts the fact $\delta(G) \geq 5$ for any planar graph. Proof of the theorem is completed. Next, we only consider the case $\Delta \leq M \leq \Delta + 2$.

Claim 1. $\delta \geq M - \Delta + 3$.

Proof. If there is some $e = uv \in E(G)$ such that $d(v) \leq M - \Delta + 2$, then $d(u) + d(v) \leq \Delta + (M - \Delta + 2) \leq M + 2$ and $d(v) \leq 5 \leq \left\lfloor \frac{M+2d-1}{2d} \right\rfloor$ as $\left\lfloor \frac{M+2d-1}{2d} \right\rfloor \geq 6$, a contradiction to (C3). \blacksquare

Let v be a k -vertex of G .

(a) If $3 \leq k \leq 5$, then $w'(v) = w(v) + \sum_{k \leq i \leq 5} 1 = (k - 6) + (6 - k) = 0$ by (C4) and rule (R1);

(b) If $6 \leq k \leq M - 3$, then for all $u \in N(v)$, $d(u) \geq 6$ by (C3). Therefore, v neither receives nor gives any charge by our rule, which implies that $w'(v) = w(v) = k - 6 \geq 0$;

(c) If $M - 2 \leq k \leq \Delta$.

Case 1. $M = \Delta + 2$. Then $\delta \geq 5$ by Claim 1. For $k = \Delta$, $w'(v) \geq w(v) - 3 = \Delta - 9 = M - 11$ by (C5) and (R1).

Case 2. $M = \Delta + 1$. Then $\delta \geq 4$ by Claim 1. For $k = \Delta - 1$, $w'(v) \geq w(v) - 3 = \Delta - 1 - 6 - 3 = M - 11$ by (C5) and rule (R1). For $k = \Delta$, $w'(v) \geq w(v) - 3 - 2 = \Delta - 6 - 3 - 2 = M - 12$ by (C5) and rule (R1).

Case 3. $M = \Delta$. Then $\delta(G) \geq 3$ by Claim 1. For $k = \Delta - 2$, $w'(v) \geq w(v) - 3 = \Delta - 2 - 6 - 3 = M - 11$ by (C5) and rule (R1). For $k = \Delta - 1$, $w'(v) \geq w(v) - 3 - 2 = \Delta - 1 - 6 - 3 - 2 = M - 12$ by (C5) and rule (R1). For $k = \Delta$, $w'(v) \geq w(v) - 3 - 2 - 1 = \Delta - 6 - 3 - 2 - 1 = M - 12$ by (C5) and rule (R1).

For all cases above, $w'(v) \geq M - 12 > 0$ for any $d(v) \geq \Delta - 2$ as $M \geq 10d + 1 \geq 21$.

Let $X = \{x \in V(G) \mid d_G(x) \leq \lfloor \frac{M+2d-1}{2d} \rfloor\}$. By (C3), X is an independent set of vertices.

Claim 2. The number of $\left(\lfloor \frac{M+2d-1}{2d} \rfloor + 1\right)^+$ -vertex of G is at least $M - \lfloor \frac{M+2d-1}{2d} \rfloor + 3$. That is, $|V(G \setminus X)| \geq M - \lfloor \frac{M+2d-1}{2d} \rfloor + 3$.

Proof. Otherwise, let $Y = N_{x \in X}(x)$ and $B = B(X, Y)$ be the induced bipartite subgraph. For all $y \in Y$, $d_{G \setminus X}(y) \leq |Y| - 1 \leq M - \lfloor \frac{M+2d-1}{2d} \rfloor + 1$. Therefore, $d_B(y) = d_G(y) - d_{G \setminus X}(y) \geq d_G(y) + \lfloor \frac{M+2d-1}{2d} \rfloor - M - 1$, which implies B is a $\lfloor \frac{M+2d-1}{2d} \rfloor$ -alternator of G , a contradiction to Lemma 2.5. ■

Since $M \geq 10d + 1$, it follows that $M - 12 > \lfloor \frac{M+2d-1}{2d} \rfloor - 5$. Thus, $w'(v) \geq \lfloor \frac{M+2d-1}{2d} \rfloor - 5$ when $d_G(v) \geq \lfloor \frac{M+2d-1}{2d} \rfloor + 1$. Then $\sum_{x \in V} w(x) = \sum_{x \in V} w'(x) > (M - \lfloor \frac{M+2d-1}{2d} \rfloor + 3)(\lfloor \frac{M+2d-1}{2d} \rfloor - 5) \geq (2d-1) \left(\frac{M-1}{2d}\right)^2 - (10d-8) \frac{M-1}{2d} - 15 \geq -6\varepsilon$ as $M \geq \frac{d}{2d-1} \left(10d-8 + \sqrt{(10d-2)^2 - 24(2d-1)\varepsilon}\right) + 1$. Then this contradiction completes the proof. ■

Proof of Theorem 1.4 Let G be a minimal counterexample in terms of $|V(G)| + |E(G)|$ to Theorem 1.4. In this theorem, $M \geq 5d + 2$. We define the initial charge function $w(x) := d(x) - 4$ for all element $x \in V \cup F$. By Euler's formula $|V| - |E| + |F| = \varepsilon$, we have $\sum_{x \in V \cup F} w(x) = \sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (d(f) - 4) = -4\varepsilon < 0$.

The transition rules are defined as follows.

- (R1) Each 3-vertex (if exists) receives charge 1 from its 3-master.
- (R2) Each k -vertex with $5 \leq k \leq 7$ transfer charge $\frac{k-4}{k}$ to each 3-face that incident with it.
- (R3) Each 8^+ -vertex transfer charge $\frac{1}{2}$ to each 3-face that incident with it.

Analogous with Claim 1 in the proof of Theorem 1.3, it is easy to prove that $\delta(G) \geq 3$ when $\Delta = M$ and $\delta(G) \geq 4$ otherwise. Let v be a k -vertex of G .

For $k = 3$, then $w'(v) = w(v) + 1 = 3 - 4 + 1 = 0$ since it receives 1 from its 3-master;

For $k = 4$, then $w'(v) = w(v) = 0$ since we never change the charge by our rules;

For $5 \leq k \leq 7$, then $w'(v) \geq w(v) - k \frac{k-4}{k} = 0$ by (R2);

For $8 \leq k \leq M - 1$, then $w'(v) \geq w(v) - k \frac{1}{2} \geq 0$ by (R3);

If $M > \Delta$, then $M - 1 \geq \Delta$. Thus $w(v) \geq 0$ for all $v \in V(G)$. Otherwise, $\Delta = M$. Then for $k = \Delta$, $w'(v) \geq w(v) - \frac{1}{2}M - 1 = \frac{M}{2} - 5$ by (C5) and rules (R1), (R3). Since $M \geq 5d + 2 \geq 12$, we have $w'(v) \geq \frac{M}{2} - 5 > 0$.

Let f be a k -face of G .

If $k \geq 4$, then $w'(f) = w(f) \geq 0$ since we never change the charge of them by our rules;

If $k = 3$, assume that $f = [v_1, v_2, v_3]$ with $d(v_1) \leq d(v_2) \leq d(v_3)$. It is easy to see $w(f) = -1$. Consider the subcases as follows.

(a) Suppose $d(v_1) = 3$. Then $M = \Delta$ and $d(v_2) = d(v_3) = \Delta$ by (C3). Thus, $w'(f) = w(f) + \frac{1}{2} \times 2 = 0$ by (R3);

(b) Suppose $d(v_1) = 4$. Then $d(v_3) \geq d(v_2) \geq M - 2d + 4 - d(v_1) \geq 3d + 2 \geq 8$ by (C2). Therefore, $w'(f) = w(f) + \frac{1}{2} \times 2 = 0$ by (R3);

(c) Suppose $d(v_1) = 5$. Then $d(v_3) \geq d(v_2) \geq M - 2d + 4 - d(v_1) \geq 3d + 1 \geq 7$ by (C2). Therefore, $w'(f) = w(f) + \frac{3}{7} \times 2 + \frac{1}{5} > 0$ by (R2).

(d) Suppose $d(v_1) = m \geq 6$. Then $d(v_3) \geq d(v_2) \geq 6$. Therefore, $w'(f) \geq w(f) + 3 \times \min\{\frac{m-4}{m}, \frac{1}{2}\} = 0$ by (R2) and (R3).

Thus, we have $\sum_{x \in V \cup F} w'(x) \geq 0$ which is a contradiction with $\sum_{x \in V \cup F} w'(x) = \sum_{x \in V \cup F} w(x) < 0$. ■

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